Define

$$h^*(t,\bar{Z}(t-)) = \left\{ W(t) - \frac{E(W(t)Y(t)I(C>t)\frac{\bar{G}^*(t|Z)}{\bar{G}(t|Z)}\exp(\beta W(t)))}{E(Y(t)I(C>t)\frac{\bar{G}^*(t|Z)}{\bar{G}(t|Z)}\exp(\beta W(t)))} \right\} \bar{G}^*(t \mid Z),$$

where $\bar{G}^*(t \mid Z)$ is a conditional survivor function of C, given Z, corresponding with a hazard $\lambda_C(t \mid \bar{Z}(t-))$ approximating the true conditional survivor function $\bar{G}(t \mid X)$ but restricted to only depend on the covariates Z entering the full data multiplicative intensity model. Firstly, we note that $g(h^*) = 0$, which proves that $\int h^*(t,\bar{Z}(t-))dM(t)$ is an element of our class of full data estimating functions (3.9). Secondly, note that $IC_{02}(Y \mid G, D_{h^*}(\cdot \mid \mu,\rho))$ equals (3.11) if in truth $\bar{G}(t \mid X) = \bar{G}^*(t \mid Z)$ (i.e., our IPCW-estimating function reduces to the optimal estimating function under the assumption that $\lambda_C(t \mid X) = \lambda_C(t \mid Z)$). We propose to estimate h^* by substitution of estimators G_n^* of G^* and G_n of G, and by estimating the empirical expectations. One can base G_n^* on fitting a multiplicative intensity model for the censoring process $A(t)$ only using a past $(\bar{A}(t-),\bar{Z}(t-))$ while one can use the entire past for G_n. Let h_n^* be the resulting estimate of h^*. With this choice of h_n^*, the estimator β_n^0 is at least as efficient as the usual partial likelihood estimator if in truth $\lambda_C(t \mid X) = \lambda_C(t \mid Z)$ and remains consistent and asymptotically normal in model $\mathcal{M}(\mathcal{G})$. Therefore, this estimator is a true generalization of the usual partial likelihood estimator of β that ignores the observed covariates beyond $\bar{Z}(t)$. This estimator was analyzed by Robins (1993a) (for $N(t) = I(T \le t)$ being a failure time counting process), and used to analyze an AIDS trial in Robins and Finkelstein (2000). Pavlic, van der Laan, Buttler (2002) used this estimator in general multiplicative intensity models to analyze a recurrent event data set.

A nice fact is that β_n^0 can be implemented by using the weight option in the S-plus function Coxph() and assigning to each observation line (corresponding with a time-point t at which covariates change or events occur) of a subject the weight $w(t) \equiv I(C>t)\bar{G}_n^*(t \mid Z)/\bar{G}_n(t \mid X)$. The reason that the weighting still works for the baseline hazard is that at the true β

$$\lambda_0(t) = \lambda_0(t \mid \beta) = \frac{E(dN(t)w(t))}{E(Y(t)\exp(\beta W(t)w(t))}.$$

If $G_n = G$, then our general asymptotic Theorem 2.4 in Chapter 2 shows that, under regularity conditions, this estimator β_n^0 will be asymptotically linear with influence curve

$$c(\beta)^{-1} IC_0(Y \mid G, D_h(\cdot \mid \beta,\rho)), \tag{3.12}$$

where $c(\beta) = -\frac{d}{d\beta} E IC_0(Y \mid G, D_h(\cdot \mid \beta,\rho))$ and h is the limit of h_n^*. If G is estimated according to a CAR model (such as the multiplicative intensity model for $A(t) = I(C \le t)$ w.r.t. $\mathcal{F}(t)$) whose unspecified parameters generate a tangent space $T_2(P_{F_X,G})$ in the observed data model, then Theorem 2.4 shows that β_n^0 will be asymptotically linear with influence curve equal to the projection of (3.12) onto the orthogonal complement of $T_2(P_{F_X,G})$. In particular, this means that one could use (3.12) as a "conservative" influence curve to construct conservative confidence intervals.

To find the correct influence curve of β_n^0, one has to calculate the projection operator onto $T_2(P_{F_X,G})$ that is provided in Lemma 3.2 in the next subsection for the Cox proportional hazards model for $\lambda_C(t \mid X)$. Application of Lemma 3.2 (i.e., (3.17)) teaches us that if $IC_0(Y \mid G, D_h) = D_h(X)\Delta/\bar{G}(T \mid X)$ and $\lambda_C(t \mid X)$ is modeled with the Cox proportional hazards model $\lambda_C(t \mid X) = \lambda_{0A}(t)\exp(\alpha W_A(t))$, then the influence curve $IC(Y)$ of β_n^0 is given by

$$\begin{aligned} IC(Y) &= c(\beta)^{-1}\left\{IC_0(Y \mid G, D_h(\cdot \mid \beta,\rho)) - E(IC_0 S_\alpha^\top)E(S_\alpha^* S_\alpha^\top)^{-1}S_\alpha\right\} \\ & c(\beta)^{-1}\left\{+\int \frac{E(D_h(X)I(T>t)\exp(\alpha W_A(t))}{E(Y_A(t)\exp(\alpha W_A(t))} dM_G(t)\right\}, \end{aligned}$$

where

$$dM_G(t) = dA(t) - E(dA(t) \mid \mathcal{F}(t)) = dA(t) - Y_A(t)\lambda_{0A}(t)\exp(\alpha W_A(t)),$$

and $S_\alpha = \int W_A(t)dM_G(t)$ is the partial likelihood score of α. Here $Y_A(t) = I(C \ge t, T \ge t)$ is the indicator of $A(t)$ being at risk of jumping at time t, given the observed past. The numerator can be estimated with inverse probability of censoring weighting by noticing that $E(D_h(X)I(T > t)\exp(\alpha W_A(t)) = E(D_h(X)I(T>t)\exp(\alpha W_A(t)I(C>T)/\bar{G}(T \mid X))$. Notice that it is straightforward to estimate $IC(Y)$. Let $\widehat{IC}(Y)$ be the estimated influence curve. We can now estimate the covariance matrix of the normal limiting distribution of $\sqrt{n}(\beta_n^0 - \beta)$ with

$$\hat{\Sigma} = \frac{1}{n}\sum_{i=1}^n \widehat{IC}(Y_i)\widehat{IC}(Y_i)^\top.$$

Similarly, we can apply Lemma 3.2 to compute the influence curve for the case where $IC_0(Y \mid G, D_h) = IC_{02}(Y \mid G, D_h)$ is given by $\int\{h(t,\bar{Z}(t-)) - g(h)(t)\}\frac{I(C>t)}{\bar{G}(t|X)}dM(t)$. In this case, the influence curve of β_n^0 is given by

$$\begin{aligned} IC(Y) &= c(\beta)^{-1}\left\{IC_{02}(Y \mid G, D_h(\cdot \mid \beta,\rho)) - E(IC_{02}S_\alpha^\top)E(S_\alpha S_\alpha^\top)^{-1}S_\alpha\right\} \\ & +c(\beta)^{-1}\int \frac{E\left\{\int_t^\infty h'(u)dM(u)\frac{I(T\ge t)\exp(\alpha W_A(t))\Delta}{\bar{G}(T|X)}\right\}}{E\left\{Y_A(t)\exp(\alpha W_A(t))\right\}} dM_G(t), \end{aligned}$$

where $h'(u) \equiv h(u) - g(h)(u)$. This finishes the methodology for β_n^0 since we defined the estimator, and provided its influence curve and corresponding confidence interval.

The optimal mapping from full data estimating functions to observed data estimating functions is obtained by subtracting the projection of $IC_0(Y)$ onto T_{CAR} as given in the next section and Theorem 1.1 in Chapter