$$-\int \frac{E\left\{D_h(X)I(T \geq t)\exp(\beta_A W(t))\frac{I(C\geq t)}{\bar{G}(t|X)}\right\}}{E\{Y_A(t)\exp(\beta_A W(t))\}} dM_G(t) \quad (3.17)$$

(Note that we can set $\Delta/\bar{G}(T \mid X)$ to 1 in this formula again).

As remarked earlier, since C can be discrete on an arbitrarily fine grid, the projection (3.16) for the *discrete* multiplicative intensity model also holds for the continuous multiplicative intensity model under appropriate measurability conditions. For a formal treatment of the continuous case, see van der Vaart (2001). In Chapter 6 we state an immediate generalization (Lemma 6.1) of this lemma to multiplicative intensity models for a general counting process $A(t)$.

Proof. The statements up to and including (3.15) were proved in Section 2.2 of Chapter 2. For the projection onto T_{CAR}, we refer to Theorem 1.1, which was only formally proved for the case where C is discrete. Thus, we only need to show the last projection result (3.17) and the general formulas $g(H_{V,1})(t)$ and $g(H_{V,2})(t)$. In this proof, we will now and then suppress the index A that we used for the parameters of the censoring intensity $E(dA(t) \mid \mathcal{F}(t))$. Firstly, consider the special case $V(C,X) = U_G(D_h)(Y) = D_h(X)I(C > T)/\bar{G}(T \mid X)$. Since $H_{V,1} = 0$, we have

$$H_V(t, \mathcal{F}(t)) = -E(D_h(X)\Delta/\bar{G}(T \mid X) \mid C > t, \bar{X}(t)).$$

Let us denote the expectation with $f(\bar{X}(t))$. Thus, the numerator of $-g(H_V)(t)$ is given by $E(E(D_h(X)\Delta/\bar{G}(T \mid X) \mid C > t, \bar{X}(t))Y_A(t)\exp(\alpha W(t)))$. We have $E(f(\bar{X}(t))Y_A(t)\exp(\alpha W(t))) = E(f(\bar{X}(t))I(T \geq t)\bar{G}(t \mid X)\exp(\alpha W(t)))$, where we use $Y_A(t) = I(C \geq t, T \geq t)$. We can move $I(T \geq t)\bar{G}(t \mid X)\exp(\alpha W(t))$ inside the conditional expectation of $f(\bar{X}(t))$. The resulting term can now be rewritten as follows:

$$E(E(D_h(X)\Delta\bar{G}(t \mid X)I(T \geq t)\exp(\alpha W(t))/\bar{G}(T \mid X) \mid C > t, \bar{X}(t)))$$

$$= E(E(D_h(X)I(T \geq t)\exp(\alpha W(t)) \mid C > t, \bar{X}(t)))$$

$$= E(E(D_h(X)I(T \geq t)\exp(\alpha W(t)) \mid \bar{X}(t)))$$

$$= E(D_h(X)I(T \geq t)\exp(\alpha W(t))).$$

At the first equality, we conditioned on X and $C > t$ and used that $E(\Delta \mid C > t, X) = \bar{G}(T \mid X)/\bar{G}(t \mid X)$, and at the second equality we use that, by CAR, for any function $V(X)$ $E(V(X) \mid C > t, \bar{X}(t)) = E(V(X) \mid \bar{X}(t))$. Finally, for the purpose of estimation of this last expectation we note that

$$E(D_h(X)I(T \geq t)\exp(\alpha W(t))) = E(D_h(X)\exp(\alpha W(t))\Delta/\bar{G}(T \mid X)),$$

which proves (3.17).

For a general choice of $V = V(X,C)$ we have $g(H_V) = g(H_{V,1}) - g(H_{V,2})$, where $H_{V,1} = E(V(X,C) \mid C = t, \bar{X}(t))$ and $H_{V,2} = E(V(X,C) \mid C > t, \bar{X}(t))$. The numerator $E(E(V(X,C) \mid C > t, \bar{X}(t))Y_A(t)\exp(\alpha W(t)))$ of $g(H_{V,2})$ is given by

$$E(E(V(X,C)I(T \geq t)\bar{G}(t \mid X)\exp(\alpha W(t)) \mid C > t, \bar{X}(t))),$$

Let $Q_{V,2}(\bar{X}(t)) = E(V(X,C)I(T \geq t)\bar{G}(t \mid X)\exp(\alpha W(t)) \mid C > t, \bar{X}(t))$. Then the last term is given by

$$E(Q_{V,2}(\bar{X}(t))I(C \geq \min(t,T))/\bar{G}(\min(t,T) \mid X)).$$

The numerator $E(E(V(X,C) \mid C = t, \bar{X}(t))Y_A(t)\exp(\alpha W(t)))$ of $g(H_{V,1})$ is given by

$$EQ_{V,2}(\bar{X}(t)) = E(E(V(X,t)\bar{G}(t \mid X)I(T \geq t)\exp(\alpha W(t)) \mid C = t, \bar{X}(t))).$$

By CAR, we have for any function $D(X)$ that $E(D(X) \mid C = t, \bar{X}(t)) = E(D(X) \mid \bar{X}(t))$. Thus, the numerator $E(V(X,t)\bar{G}(t \mid X)I(T \geq t)\exp(\alpha W(t)))$ of $g(H_{V,2})$ is actually given by

$$E(V(X,t)\bar{G}(t \mid X)I(T \geq t)\exp(\alpha W(t))I(C \geq \min(t,T))/\bar{G}(\min(t,T) \mid X)).$$

□

3.4 Optimal Mapping into Estimating Functions

The mapping from D_h into observed data estimating functions is a sum of two mappings IC_0 and IC_{CAR}, where IC_0 is an initial mapping satisfying $E_G(IC_0(Y \mid G, G) \mid X) = D(X)$ F_X-a.e. for D in a non empty subset $\mathcal{D}(\rho_1(F_X), G) \subset \mathcal{D}$ and IC_{CAR} is the projection of IC_0 onto the tangent space $T_{CAR}(P_{F_X,G})$. By Theorem 1.1, we have

$$T_{CAR}(P_{F_X,G}) = \overline{\left\{\int Q(u, \bar{X}(u))dM_G(u) : Q\right\}} \subset L_0^2(P_{F_X,G}),$$

where

$$dM_G(u) = I(\tilde{T} \in du, \Delta = 0) - I(\tilde{T} \geq u)\Lambda_C(du \mid X)$$

is the martingale of $A(\cdot) = I(C \leq \cdot)$ w.r.t. history $\mathcal{F}(t) = \sigma(\bar{A}(t-), \bar{X}(\min(C, t-)))$. Here $Q(u, \bar{X}(u))$ ranges over all functions for which $\int Q(u, \bar{X}(u))dM_G(u)$ has finite variance so that it is an element of $L_0^2(P_{F_X,G})$. By Theorem 1.1 (for the discrete case) and van der Vaart (2001) (for the continuous case), we also have for any $D \in \mathcal{D}$

$$IC_{CAR}(Y \mid Q(F_X, G), G, D) = \int Q(F_X, G)(u, \bar{X}(u))dM_G(u),$$